Rank-one perturbation formulas for the planar simple random walk in random potentials (Stochastic Analysis on Large Scale Interacting Systems)

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By

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Abstract

In this paper, we survey fluctuations around the mean value of the travel cost for the simple random walk in random potentials on the multidimensional cubic lattice. To analyze the fluctuations, it is important to obtain an upper bound on how much travel cost may change when a potentials are changed. Zerner [12, Lemma 12] studied this as a rank-one perturbation formula, and it works very well if the dimension is greater than two. We introduce a new rank-one perturbation formula toward improving the results in the two dimensional case.

§ 1. Introduction

Let $(S_k)_{k=0}^{\infty}$ be the simple random walk on the d-dimensional cubic lattice $\mathbb{Z}^d$, $d \geq 2$. For $x \in \mathbb{Z}^d$, write P^x for the law of the random walk starting at x, and E^x for the associated expectation. Furthermore, we consider the measurable space $\Omega := [0, \infty)^{\mathbb{Z}^d}$ endowed with the canonical σ-field $\mathcal{G}$. Let $\mathbb{P}$ be the corresponding product measure on $(\Omega, \mathcal{G})$ and denote an element of Ω by $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$, which is called the potential. To avoid trivialities we suppose that $\omega(0)$ is not almost surely equal to 0.

For $y \in \mathbb{Z}^d$, $H(y)$ stands for the hitting time of $(S_k)_{k=0}^{\infty}$ to y, i.e., $H(y) := \inf\{k \geq 0; S_k = y\}$. Furthermore we define for $x, y \in \mathbb{Z}^d$,

$$e(x, y, \omega) := E^x \left[\exp\left\{ - \sum_{k=0}^{H(y)-1} \omega(S_k) \right\} \mathbf{1}_{\{H(y) < \infty\}} \right], \tag{1.1}$$

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where $e(x, y, \omega) := 1$ if $x = y$. Let us now introduce the travel cost $a(x, y, \omega)$ from x to y for the simple random walk in a potential ω as follows:

$$a(x, y, \omega) := -\log e(x, y, \omega), \qquad x, y \in \mathbb{Z}^d.$$

Throughout this paper, we drop ω in the notation if there is no confusion.

Notice that the subadditivity

$$a(x, z) \le a(x, y) + a(y, z), \qquad x, y, z \in \mathbb{Z}^d,$$

immediately follows from the strong Markov property, see the proof of Proposition 2 of [12]. As we are working with i.i.d. potentials, the subadditive ergodic theorem shows the following proposition. This means that $a(0, nx)$ grows roughly linearly as $n \to \infty$ along the direction x, see the proof of [12, Proposition 4] for more details.

Proposition 1.1 (Zerner). *Assume $\mathbb{E}[\omega(0)] < \infty$. Then there exists a norm $\alpha(\cdot)$ on $\mathbb{R}^d$ (which is called the Lyapunov exponent) such that for all $x \in \mathbb{Z}^d$, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$,*

$$\lim_{n \to \infty} \frac{1}{n} a(0, nx) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[a(0, nx)] = \inf_{n \ge 1} \frac{1}{n} \mathbb{E}[a(0, nx)] = \alpha(x).$$

Furthermore, $\alpha(\cdot)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, and satisfies

$$-\log \mathbb{E}[e^{-\omega(0)}] \le \frac{\alpha(x)}{\|x\|_1} \le \log(2d) + \mathbb{E}[\omega(0)],$$

where $\| \cdot \|_1$ is the ℓ^1-norm on $\mathbb{R}^d$.

Remark 1.2. The present paper always assumes that potentials have at least second moment (see hypotheses (A1) and (A2) below), so that, for simplicity, we can use the result obtained by Zerner. However, it is known that the above proposition is valid under lower moments. In fact, Mourrat [9, Theorems 1.1] proved the following: If Z is the minimum of $2d$ i.i.d. random variables distributed as $\omega(0)$, then for each $x \in \mathbb{Z}^d$,

$$\mathbb{E}[Z] < \infty \text{ if and only if } \frac{1}{n} a(0, nx) \text{ converges a.s.}$$

§ 2. Mean value fluctuations for the travel cost

Let us first introduce the following assumptions for the potential:

(A1) $\mathbb{E}[e^{\gamma \omega(0)}] < \infty$ for some $\gamma > 0$.

(A2) $\mathbb{E}[\omega(0)^2] < \infty$.

(A3) The law of $\omega(0)$ has strictly positive support.

Assuming some of these conditions, we have the exponential concentration for the upper tail and the Gaussian concentration for the lower tail.

Theorem 2.1. *Assume (A1). In addition, suppose that (A3) is valid if $d = 2$. Then, there exist constants $0 < C_1, C_2 < \infty$ such that for all large $x \in \mathbb{Z}^d$ and for all $t \geq 0$,*

$$\mathbb{P}\big(a(0,x) - \mathbb{E}[a(0,x)] \geq t\|x\|_1^{1/2}\big) \leq C_1 e^{-C_2 t}.$$

Theorem 2.2. *Assume (A2). In addition, suppose that (A3) is valid if $d = 2$. Then, there exists a constant $0 < C_3 < \infty$ such that for all large $x \in \mathbb{Z}^d$ and for all $t \geq 0$,*

$$\mathbb{P}\big(a(0,x) - \mathbb{E}[a(0,x)] \leq -t\|x\|_1^{1/2}\big) \leq e^{-C_3 t^2}.$$

We omit the proofs since these are too long to reproduce in this paper, and refer the reader to [8]. Instead, let us survey the proof of the following proposition obtained by Zerner [12, Theorem 11] in the end of this section. This is because the proof of it is simple, and is done by using the essential argument and a key tool to obtain Theorems 2.1 and 2.2. Note that the following proposition immediately follows from Theorems 2.1 and 2.2 under the same assumption of Theorem 2.1.

Proposition 2.3 (Zerner). *Assume (A2). Furthermore, suppose that (A3) is valid if $d = 2$. Then, there exists a constant $0 < C_4 < \infty$ such that*

$$\mathrm{Var}(a(0,y)) \leq C_4 \|y\|_1, \qquad y \in \mathbb{Z}^d. \tag{2.1}$$

In the statements of the above theorems and proposition, we suppose the additional assumption (A3) in $d = 2$, but it may not be necessary. In fact, Zerner expects that in $d = 2$, Proposition 2.3 holds without (A3), see below Theorem 11 of [12]. This conjecture may not be applied to Theorems 2.1 and 2.2 since they need more precise arguments. However, it can be an important common problem in fluctuations of the travel cost.

Furthermore, the above proposition derives the following generalization of Proposition 1.1, which is called the uniform shape theorem [12, Theorem 13].

Proposition 2.4 (Zerner). *Suppose that $\mathbb{E}[\omega(0)^d] < \infty$. In addition, we assume (A3) if $d = 2$. Then, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \frac{a(x_n, y_n)}{\alpha(x_n - y_n)} = 1.$$

for all sequence $x_n, y_n \in \mathbb{Z}^d$ *such that* $c(\|x_n\|_1 \vee \|y_n\|_1) \leq \|x_n - y_n\|_1 \to \infty$ *as* $n \to \infty$ *for some* $c > 0$.

The key tool mentioned above is the following rank-one perturbation formula obtained by Zerner [12, Lemma 12].

Proposition 2.5 (Zerner). *Let* $z \in \mathbb{Z}^d$ *and* $\omega_1, \omega_2 \in \Omega$ *such that* $\omega_1(x) = \omega_2(x)$ *for* $x \neq z$ *and* $\omega_1(z) \leq \omega_2(z)$. *Then,* $a(0, y, \omega_2) - a(0, y, \omega_1)$ *is nonnegative, and is bounded from above by the minimum of*

$$-\log Q^{0,x}_{\omega_1}(H(y) \leq H(z))$$

and

$$\omega_2(z) + \big(1 - \min\{e^{-\omega_1(z)}, P^0(H_2(0) < \infty)\}\big)^{-1}, \tag{2.2}$$

where $H_2(0)$ *is the time of the second visit of* 0 *for the random walk, and* $Q^{0,x}_{\omega}$ *is the probability measure such that*

$$\frac{dQ^{0,x}_{\omega}}{dP^0} = e(0, x, \omega)^{-1} \exp\Bigg\{-\sum_{k=0}^{H(x)-1} \omega(S_k)\Bigg\} \mathbf{1}_{\{H(x)<\infty\}}.$$

For the proofs of Theorem 2.1, Theorem 2.2 and Proposition 2.3, it is important that the last term in (2.2) is finite. If $d \geq 3$ then this is trivial since the simple random walk is transient, i.e., $P^0(H_2(0) < \infty) < 1$. On the other hand, $P^0(H_2(0) < \infty) = 1$ holds for $d = 2$. This means that the last term in (2.2) is equal to $\omega_2(y) + (1 - e^{-\omega_1(y)})^{-1}$, which is not finite if $\omega_1(y) = 0$. Additional assumption (A3) always guarantees the finiteness for $d = 2$.

With these observations, it is meaningful to derive a new rank-one perturbation formula for $d = 2$ without additional assumption (A3). In the next section, we try to do this. Unfortunately there still remains an obstacle to improve Proposition 2.3 to $d = 2$, see the end of Section 3.2.

We close this section with the proof of Proposition 2.3 and several remarks.

Proof of Proposition 2.3. Let us enumerate $\mathbb{Z}^d = \{x_1, x_2, \dots\}$. Then, $\mathcal{F}_0$ denotes the trivial σ-field and let $\mathcal{F}_i$ be the σ-field generated by $\omega(x_1), \dots, \omega(x_i)$. Moreover, define for $i \geq 1$,

$$\Delta_i := \mathbb{E}[a(0, y)|\mathcal{F}_i] - \mathbb{E}[a(0, y)|\mathcal{F}_{i-1}].$$

Since $\mathbb{E}[a(0, y)|\mathcal{F}_i]$, $i \geq 0$, is a martingale with respect to the filtration $\mathcal{F}_i$, $i \geq 0$, we have

$$\mathrm{Var}(a(0, y)) = \sum_{i=1}^{\infty} \mathbb{E}[\Delta_i^2].$$

Note that we can write

$$\Delta_i(\omega) = \int_\Omega \{a(0, y, [\omega, \sigma]_i) - a(0, y, [\omega, \sigma]_{i-1})\} \, \mathbb{P}(d\sigma),$$

where $[\omega, \sigma]_0 := \sigma$ and

$$[\omega, \sigma]_i := (\omega(x_1), \dots, \omega(x_i), \sigma(x_{i+1}), \dots), \qquad i \geq 1.$$

Schwarz's inequality yields that

$$\begin{aligned} &\mathrm{Var}(a(0, y)) \\ &\leq \sum_{i=1}^\infty \int_\Omega \int_\Omega \{a(0, y, [\omega, \sigma]_i) - a(0, y, [\omega, \sigma]_{i-1})\}^2 \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &= 2\sum_{i=1}^\infty \int_\Omega \int_\Omega \{a(0, y, [\omega, \sigma]_i) - a(0, y, [\omega, \sigma]_{i-1})\}^2 \mathbf{1}_{\{\sigma(x_i) \leq \omega(x_i)\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega), \end{aligned} \tag{2.3}$$

where we used the symmetry of the integrand. Proposition 2.5 proves that there exists a constant $0 < C_5 < \infty$ such that the most right side of the above is bounded from above by

$$\begin{aligned} 2\sum_{i=1}^\infty &\biggl(\int_\Omega \int_\Omega (\omega(x_i) + C_5)^2 \mathbf{1}_{\{Q^{0,y}_{[\omega,\sigma]_{i-1}}(H(y) < H(x_i)) < 1/2\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &+ \int_\Omega \int_\Omega \{\log Q^{0,y}_{[\omega,\sigma]_{i-1}}(H(y) < H(x_i))\}^2 \mathbf{1}_{\{Q^{0,y}_{[\omega,\sigma]_{i-1}}(H(y) < H(z)) \geq 1/2\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \biggr). \end{aligned}$$

By using $(\log t)^2 \leq 1 - t$ for $1/2 \leq t \leq 1$, this is smaller than or equal to

$$\begin{aligned} &2\sum_{i=1}^\infty (2\mathbb{E}[(\omega(0) + c)^2] + 1)\mathbb{P}\bigl(Q^{0,y}_\omega(H(x_i) < H(y))\bigr) \\ &= 2(2\mathbb{E}[(\omega(0) + c)^2] + 1) \, \mathbb{E}\Bigl[E_{Q^{0,y}_\omega}[\#\mathcal{A}]\Bigr], \end{aligned}$$

where $\mathcal{A} := \{S_k ; 0 \leq k < H(y)\}$. Lemma 3 of [12] guarantees that $\mathbb{E}[E_{Q^{0,y}_\omega}[\#\mathcal{A}]] \leq C_6 \|y\|_1$ for some constant $0 < C_6 < \infty$, and therefore (2.1) follows. □

Remark 2.6. In the case where the law of potentials has bounded and strictly positive support, Ioffe–Velenik [6, Lemma 4] and Sodin [10, Theorem 1] proved the Gaussian concentration for both tails: There exists a constant $0 < c < \infty$ such that for all sufficiently large $x \in \mathbb{Z}^d$ and for all $t \geq 0$,

$$\mathbb{P}\bigl(|a(0, x) - \mathbb{E}[a(0, x)]| \geq t\|x\|_1^{1/2}\bigr) \leq e^{-ct^2}.$$

In this context, Sznitman [11, Theorem 2.1] proved exponential concentrations for Brownian motion in a truncated Poissonian potential. This model is a continuous counterpart of the simple random walk in random potentials. With these observations, there are few results for unbounded nonnegative potentials, and Theorems 2.1 and 2.2 extend the aforementioned previous works to concentrations for unbounded nonnegative potentials.

Remark 2.7. Sodin [10, Theorem 2] proved that we have the sublinear variance bound if $\mathbb{P}(\omega(0) = a) = \mathbb{P}(\omega(0) = b) = 1/2$ for some $0 < a < b$, i.e., there exists a constant $0 < c < \infty$ such that

$$\mathrm{Var}(a(0,x)) \leq \frac{c\|x\|_1}{\log(\|x\|_1 + 2)}.$$

It is expected that this is valid under the assumption of Theorem 2.1 or Theorem 2.2. To this end, we may apply the approach taken in [3] for the first passage percolation on $\mathbb{Z}^d$. A rank-one perturbation formula also plays a key role in [3], and it seems to be a sharper estimate than Proposition 2.5 for our model. After all, we need to review the rank-one perturbation formula more carefully.

§ 3. On a rank-one perturbation formula for $d = 2$

Throughout this section, let $d = 2$ and we always assume the following hypothesis, which is weaker than (A3):

(A4) $\mathbb{P}(\omega(0) = 0) < p_c$, where p_c (≈ 0.592) is the critical threshold for site percolation on $\mathbb{Z}^2$.

In this section, we will derive a new rank-one perturbation formula under assumption (A4) following the approach taken in [9, Section 6]. Then one tries to apply it to the argument taken in the proof of Proposition 2.3. As mentioned above, our method is not enough to answer Zerner's conjecture, so we will point out its fault in the end of Subsection 3.2.

§ 3.1. Preliminary

In this subsection, we prepare some notation to derive a new rank-one perturbation formula. Let $\|\cdot\|_\infty$ be the ℓ^∞-norm on $\mathbb{R}^d$. For any two points $x, y \in \mathbb{Z}^2$, write $x \sim y$ or $x \overset{*}{\sim} y$ if $\|x - y\|_1 = 1$ or $\|x - y\|_\infty = 1$, respectively. A path $r = (r_0, \dots, r_l)$ is $\mathbb{Z}^2$- or $*$-connected if $r_i \sim r_{i+1}$ or $r_i \overset{*}{\sim} r_{i+1}$ for all $0 \leq i \leq l - 1$, respectively. Moreover, a subset $A \subset \mathbb{Z}^2$ is said to be $\mathbb{Z}^2$- or $*$-connected if for any two sites $x, y \in A$, there exists a $\mathbb{Z}^2$- or $*$-connected path from x to y, respectively. For a $*$-connected nonempty set A, we define its exterior boundary as

$$\partial_{\mathrm{ext}} A := \left\{ x \in \mathbb{Z}^d \setminus A; \begin{matrix} x \overset{*}{\sim} y \text{ for some } y \in A, \text{ and there exists a } \mathbb{Z}^2\text{-path} \\ \text{from } x \text{ to } \infty \text{ without using any site of } A \end{matrix} \right\}$$

By assumption (A4), we can take $\kappa > 0$ satisfying $\mathbb{P}(\omega(0) < \kappa) < p_c$. A site $x \in \mathbb{Z}^d$ is said to be bad if $\omega(x) < \kappa$, and good otherwise. $\mathbb{Z}^2$- or $*$-connected path $r = (r_0, \dots, r_l)$ is called bad (resp. good) if each site r_i is bad (resp. good). For $x \in \mathbb{Z}^2$ let $\mathcal{C}_x$ be a $*$-connected bad cluster containing a bad site x, i.e., the set of all sites connected to x by a bad $*$-connected path. It is well known from [5, Theorem 6.1] that there exist constants $0 < C_7, C_8 < \infty$ such that for all $x \in \mathbb{Z}^2$ and $t \geq 0$,

$$\mathbb{P}(\#\mathcal{C}_x \geq t) \leq C_7 e^{-C_8 t}. \tag{3.1}$$

Let $\overline{\mathcal{C}}_x := \mathcal{C}_x \cup \partial_{\mathrm{ext}} \mathcal{C}_x$, and we decide $\overline{\mathcal{C}}_x = \partial_{\mathrm{ext}} \mathcal{C}_x = \{x\}$ if $\mathcal{C}_x = \emptyset$. Note that any site of $\partial_{\mathrm{ext}} \mathcal{C}_x$ is good. Furthermore, Lemma 2.23 in [7] guarantees that $\partial_{\mathrm{ext}} \mathcal{C}_x$ is $\mathbb{Z}^2$-connected.

Finally, define the distance $\mathrm{dist}(A, B)$ between two subsets $A, B \subset \mathbb{R}^2$ as

$$\mathrm{dist}(A, B) := \inf\{\|a - b\|_1; a \in A,\ b \in B\}.$$

In particular, write $\mathrm{dist}(a, B) := \mathrm{dist}(\{a\}, B)$ to shorten notation.

§ 3.2. A new rank-one perturbation formula

Let us introduce for $\omega \in \Omega$,

$$H^{\mathrm{g}}(y, \omega) := \inf\{k \geq 0; S_k \in \partial_{\mathrm{ext}} \mathcal{C}_y(\omega)\}.$$

Then, define $e^*(x, y)$ by replacing $H(y)$ with $H^{\mathrm{g}}(y)$ in (1.1), and set

$$e^{\mathrm{g}}(x, y) := \max_{x' \in \partial_{\mathrm{ext}} \mathcal{C}_x} e^*(x', y).$$

Moreover, let

$$a^*(x, y) := -\log e^*(x, y), \quad a^{\mathrm{g}}(x, y) := -\log e^{\mathrm{g}}(x, y).$$

The quantity $a^{\mathrm{g}}(x, y)$ means the travel cost between the two exterior boundaries $\partial_{\mathrm{ext}} \mathcal{C}_x$ and $\partial_{\mathrm{ext}} \mathcal{C}_y$.

The next proposition compares between two travel costs $a(x, y)$ and $a^{\mathrm{g}}(x, y)$. The proof follows from the same argument as in [9, Proposition 6.1], and we omit the proof.

Proposition 3.1. *For $x, y \in \mathbb{Z}^2$, we have*

$$a^{\mathrm{g}}(x, y) \leq a(x, y) \leq a^{\mathrm{g}}(x, y) + u(x) + u(y),$$

where

$$u(z) := \sum_{w \in \overline{\mathcal{C}}_z(\omega)} (\omega(w) + 2 \log 2), \qquad z \in \mathbb{Z}^2.$$

To observe how much travel cost may change when potentials are changed, it is useful to consider the following new travel costs mixed by two potentials: For $\omega_1, \omega_2 \in \Omega$,

$$\widetilde{a}^*(x, y, \omega_1, \omega_2) := -\log \widetilde{e}^*(x, y, \omega_1, \omega_2), \quad \widetilde{a}^{\mathrm{g}}(x, y, \omega_1, \omega_2) := -\log \widetilde{e}^{\mathrm{g}}(x, y, \omega_1, \omega_2),$$

where

$$\begin{aligned}\widetilde{e}^*(x, y, \omega_1, \omega_2) &:= E^x\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1)<\infty\}}\right],\\ \widetilde{e}^{\mathrm{g}}(x, y, \omega_1, \omega_2) &:= \max_{x' \in \partial_{\mathrm{ext}}\mathcal{C}_x(\omega_1)} \widetilde{e}^*(x', y, \omega_1, \omega_2).\end{aligned}$$

Finally, let us introduce a similar quantity $v(z, \omega_1, \omega_2)$ to $u(z)$ mixed by two potentials ω_1 and ω_2 as follows:

$$v(z, \omega_1, \omega_2) := \sum_{w \in \overline{\mathcal{C}}_z(\omega_1)} (\omega_2(w) + 2\log 2), \qquad z \in \mathbb{Z}^2.$$

After the preparation above, we have the next two theorems.

Theorem 3.2. *Let $z \in \mathbb{Z}^2$ and $\omega_1, \omega_2 \in \Omega$ such that $\omega_1(x) = \omega_2(x)$ for $x \neq z$ and $\omega_1(z) \leq \omega_2(z)$. Then $\widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) - a(0, y, \omega_1)$ is nonnegative and has the following upper bounds:*

1. *If $\mathrm{dist}(\overline{\mathcal{C}}_0(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) \geq 1$ and $\mathrm{dist}(\overline{\mathcal{C}}_y(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) \geq 1$, then $\widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) - a^{\mathrm{g}}(0, y, \omega_1)$ is bounded from above by the minimum of*

$$-\log \Phi_{\omega_1}^{0,y}(H^{\mathrm{g}}(y, \omega_1) < H^{\mathrm{g}}(z, \omega_1)) \tag{3.2}$$

and

$$2v(z, \omega_1, \omega_2) - \log(1 - e^{-\kappa}), \tag{3.3}$$

where

$$\begin{aligned}&\Phi_{\omega_1}^{0,y}(A)\\ &:= e^{\mathrm{g}}(0, y, \omega_1)^{-1} \max_{x \in \partial_{\mathrm{ext}}\mathcal{C}_0} E^x\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_1(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1)<\infty\}\cap A}\right].\end{aligned}$$

2. *Otherwise, one has*

$$\begin{aligned}&\widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) - a^{\mathrm{g}}(0, y, \omega_1)\\ &\leq v(0, \omega_1, \omega_2) + v(y, \omega_1, \omega_2) + 2v(z, \omega_1, \omega_2) - \log(1 - e^{-\kappa}).\end{aligned} \tag{3.4}$$

Theorem 3.3. *Suppose the same assumptions as in Theorem 3.2. If*

$$\mathrm{dist}(z, \overline{\mathcal{C}}_0(\omega_1)) \geq 1 \textit{ and } \mathrm{dist}(z, \overline{\mathcal{C}}_y(\omega_1)) \geq 1,$$

then

$$\widetilde{a}^{\mathrm{g}}(0, y, \omega_2) - \widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) = 0.$$

Otherwise, one has

$$0 \leq \widetilde{a}^{\mathrm{g}}(0, y, \omega_2) - \widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) \leq v(0, \omega_1, \omega_2) + v(y, \omega_1, \omega_2).$$

The second term of (3.3) corresponds to that of (2.2). We thus succeed in deriving the finiteness forcing the effect of potentials on bad clusters to some extent. By (3.1) the effect of $v(\cdot, \omega_1, \omega_2)$'s seems to be harmless, but the lack of the independence makes it difficult to prove (2.1). Let us first apply these theorems to the proof of Proposition 2.3 and explain this difficulty. We postpone the proofs of Theorems 3.2 and 3.3 to the next subsection.

Suppose assumptions (A2) and (A4). From (3.1) and Proposition 3.1, there exists a constant $0 < C_9 < \infty$ such that

$$\mathrm{Var}(a(0, y)) \leq \mathrm{Var}(a^{\mathrm{g}}(0, y)) + C_9.$$

Therefore, for (2.1) it suffices to estimate $\mathrm{Var}(a^{\mathrm{g}}(0, y))$. To shorten notation, let $\tau_i := [\omega, \sigma]_i$ for $i \geq 1$. Replacing $a(0, y)$ with $a^{\mathrm{g}}(0, y)$ in (2.3), one has

$$\mathrm{Var}(a^{\mathrm{g}}(0, y)) \leq 4 \sum_{i=1}^{\infty} \{J_1(i) + J_2(i)\},$$

where

$$J_1(i) := \int_{\Omega} \int_{\Omega} \{a^{\mathrm{g}}(0, y, \tau_i) - a^{\mathrm{g}}(0, y, \tau_{i-1}, \tau_i)\}^2 \mathbf{1}_{\{\sigma(x_i) \leq \omega(x_i)\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega)$$

and

$$J_2(i) := \int_{\Omega} \int_{\Omega} \{a^{\mathrm{g}}(0, y, [\omega, \sigma]_{i-1}, \tau_i) - a^{\mathrm{g}}(0, y, \tau_{i-1})\}^2 \mathbf{1}_{\{\sigma(x_i) \leq \omega(x_i)\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega).$$

We shall estimate the sum of all $J_2(i)$'s. Theorem 3.3 allows us to show that

$$\begin{aligned} \sum_{i=1}^{\infty} J_2(i) &\leq \sum_{i=1}^{\infty} \int_{\Omega} \int_{\Omega} \{v(0, \tau_{i-1}, \tau_i) + v(y, \tau_{i-1}, \tau_i)\}^2 \\ &\qquad \times \mathbf{1}_{\{\mathrm{dist}(x_i, \overline{\mathcal{C}}_0(\tau_{i-1}))=0 \text{ or } \mathrm{dist}(z, \overline{\mathcal{C}}_y(\tau_{i-1}))=0\}} \mathbf{1}_{\{\sigma(x_i) \leq \omega(x_i)\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &\leq 4 \sum_{i=1}^{\infty} (K_1(i) + K_2(i)), \end{aligned}$$

where

$$K_1(i) := \int_\Omega \int_\Omega v(0,\tau_{i-1},\tau_i)^2 \mathbf{1}_{\{\mathrm{dist}(x_i,\overline{\mathcal{C}}_0(\tau_{i-1}))=0\}} \mathbf{1}_{\{\sigma(x_i)\le\omega(x_i)\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega),$$

$$K_2(i) := \int_\Omega \int_\Omega v(0,\tau_{i-1},\tau_i)^2 \mathbf{1}_{\{\mathrm{dist}(x_i,\overline{\mathcal{C}}_y(\tau_{i-1}))=0\}} \mathbf{1}_{\{\sigma(x_i)\le\omega(x_i)\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega).$$

Schwarz's inequality implies

$$\begin{aligned}\sum_{i=1}^\infty K_1(i) &\le \sum_{i=1}^\infty \sum_{z\in\mathbb{Z}^2} \int_\Omega \int_\Omega (\tau_i(z)+2\log 2)^2 \#\overline{\mathcal{C}}_0(\tau_{i-1}) \mathbf{1}_{\{z\in\overline{\mathcal{C}}_0(\tau_{i-1})\}} \\ &\qquad \times \mathbf{1}_{\{\mathrm{dist}(x_i,\overline{\mathcal{C}}_0(\tau_{i-1}))=0\}} \mathbf{1}_{\{\sigma(x_i)\le\omega(x_i)\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &\le \sum_{i=1}^\infty \int_\Omega \int_\Omega (\omega(x_i)+2\log 2)^2 (\#\overline{\mathcal{C}}_0(\tau_{i-1}))^2 \mathbf{1}_{\{x_i\in\overline{\mathcal{C}}_0(\tau_{i-1})\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &\qquad + \sum_{i=1}^\infty \sum_{z\in\mathbb{Z}^2} \mathbb{E}\big[(\omega(z)+2\log 2)^2 \#\overline{\mathcal{C}}_0 \mathbf{1}_{\{z,x_i\in\overline{\mathcal{C}}_0\}}\big].\end{aligned} \tag{3.5}$$

Since $(\#\overline{\mathcal{C}}_0(\tau_{i-1}))^2 \mathbf{1}_{\{x_i\in\overline{\mathcal{C}}_0(\tau_{i-1})\}}$ does not depend on $\omega(x_i)$, the first term of the most right side is equal to

$$\sum_{i=1}^\infty \mathbb{E}[(\omega(0)+2\log 2)^2]\mathbb{E}\big[(\#\overline{\mathcal{C}}_0)^2 \mathbf{1}_{\{x_i\in\overline{\mathcal{C}}_0\}}\big] = \mathbb{E}[(\omega(0)+2\log 2)^2]\mathbb{E}\big[(\#\overline{\mathcal{C}}_0)^3\big],$$

which is finite by assumption (A2) and (3.1). Furthermore, thanks to Chebyshev's association inequality (see [1, Theorem 2.14]), the last term smaller than or equal to

$$\sum_{i=1}^\infty \sum_{z\in\mathbb{Z}^2} \mathbb{E}\big[(\omega(x_i)+2\log 2)^2\big]\mathbb{E}\big[\#\overline{\mathcal{C}}_0 \mathbf{1}_{\{z,x_i\in\overline{\mathcal{C}}_0\}}\big] = \mathbb{E}\big[(\omega(0)+2\log 2)^2\big]\mathbb{E}\big[(\#\overline{\mathcal{C}}_0)^3\big].$$

This is also finite by assumption (A2) and (3.1). Similarly to the above, $\sum_{i=1}^\infty K_2(i) \le C_{10}$ holds for some constant $0 < C_{10} < \infty$. With these observations, there exists a constant $0 < C_{11} < \infty$ such that

$$\sum_{i=1}^\infty J_2(i) \le C_{11}.$$

We next try to estimate the sum of all $J_1(i)$'s. Divide $J_1(i)$ into the following two terms:

$$\begin{aligned} L_1(i) + L_2(i) :=& \int_\Omega \int_\Omega \{a^{\mathrm{g}}(0,y,\tau_i) - a^{\mathrm{g}}(0,y,\tau_{i-1},\tau_i)\}^2 \\ &\qquad \times \mathbf{1}_{\{\sigma(x_i)\le\omega(x_i)\}} \mathbf{1}_{\{\text{case 2 of Theorem 3.2}\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &+ \int_\Omega \int_\Omega \{a^{\mathrm{g}}(0,y,\tau_i) - a^{\mathrm{g}}(0,y,\tau_{i-1},\tau_i)\}^2 \\ &\qquad \times \mathbf{1}_{\{\sigma(x_i)\le\omega(x_i)\}} \mathbf{1}_{\{\text{case 1 of Theorem 3.2}\}}\,\mathbb{P}(d\sigma)\mathbb{P}(d\omega).\end{aligned}$$

We use Theorem 3.2 and the same argument below (2.3) to obtain

$$\begin{aligned} L_1(i) &\leq \int_\Omega \int_\Omega \bigl(2v(x_i, \tau_{i-1}, \tau_i) - \log(1-e^{-\kappa})\bigr)^2 \\ &\quad \times \mathbf{1}_{\{\Phi^{0,y}_{\tau_{i-1}}(H^{\mathrm{g}}(y,\tau_{i-1}) < H^{\mathrm{g}}(x_i,\tau_{i-1})) < 1/2\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &\leq 2\int_\Omega \int_\Omega \bigl(2v(x_i, \tau_{i-1}, \tau_i) - \log(1-e^{-\kappa})\bigr)^2 \\ &\quad \times \Phi^{0,y}_{\tau_{i-1}}(H^{\mathrm{g}}(x_i,\tau_{i-1}) < H^{\mathrm{g}}(y,\tau_{i-1})) \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \end{aligned}$$

and

$$\begin{aligned} L_2(i) &\leq \int_\Omega \int_\Omega \bigl(\log \Phi^{0,y}_{\tau_{i-1}}(H^{\mathrm{g}}(y,\tau_{i-1}) < H^{\mathrm{g}}(z,\tau_{i-1}))\bigr)^2 \\ &\quad \times \mathbf{1}_{\{\Phi^{0,y}_{\tau_{i-1}}(H^{\mathrm{g}}(y,\tau_{i-1}) < H^{\mathrm{g}}(x_i,\tau_{i-1})) \geq 1/2\}} \, \mathbb{P}(d\sigma)\mathbb{P}(d\omega) \\ &\leq \int_\Omega \Phi^{0,y}_\omega(H^{\mathrm{g}}(x_i,\omega) < H^{\mathrm{g}}(y,\omega)) \, \mathbb{P}(d\omega). \end{aligned}$$

In view of the proof of Proposition 2.3, our task is now to prove that for some constant $0 < C_{12} < \infty$,

$$\sum_{i=1}^\infty L_1(i) + \sum_{i=1}^\infty L_2(i) \leq C_{12}\|y\|_1. \tag{3.6}$$

If we ignore the integrand $(2v(x_i, \tau_{i-1}, \tau_i) - \log(1-e^{-\kappa}))^2$ in the estimate for $L_1(i)$ above, then the following common quantity arises:

$$\mathbb{E}\Biggl[\sum_{i=1}^\infty \Phi^{0,y}_\omega(H^{\mathrm{g}}(x_i,\omega) < H^{\mathrm{g}}(y,\omega))\Biggr],$$

which is roughly bounded above by

$$\mathbb{E}\Biggl[\frac{\#\overline{\mathcal{C}}_0}{e^{\mathrm{g}}(0,y)} \max_{x \in \partial_{\mathrm{ext}}\mathcal{C}_0} E^x\Biggl[\#\Bigl(\bigcup_{v\in\mathcal{A}} \mathcal{C}_v\Bigr) \exp\Biggl\{-\sum_{k=0}^{H^{\mathrm{g}}(y)-1} \omega(S_k)\Biggr\} \mathbf{1}_{\{H^{\mathrm{g}}(y)<\infty\}}\Biggr]\Biggr].$$

If the term $\#\overline{\mathcal{C}}_0$ is harmless, this is the mean size of the union of all bad clusters encountering the random walk from 0 to y under a weighted measure. For (3.6), we have to bound this mean by const $\times \|y\|_1$, and it is a generalization of Lemma 3 in [12]. Unfortunately the author was not able to establish this. Howevr, the the techniques in [2] and [4] may be applicable and this is subject for future research.

§ 3.3. Proofs of Theorems 3.2 and 3.3

In this subsection, we shall prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. Since $\omega_1(x) \le \omega_2(x)$ for $x \in \mathbb{Z}^2$, it is clear that $\widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) - a(0, y, \omega_1)$ is nonnegative.

We first treat case 1. Let $x_1, x_2 \in \partial_{\mathrm{ext}}\mathcal{C}_0(\omega_1)$, $z_1 \in \partial_{\mathrm{ext}}\mathcal{C}_z(\omega_1)$ and let $z_2 \in \partial_{\mathrm{ext}}\mathcal{C}_z(\omega_2)$ with $e^*(z_2, y, \omega_2) = e^{\mathrm{g}}(z, y, \omega_2)$. We now divide $\widetilde{e}^*(x, y, \omega_1, \omega_2)$ into the following two parts:

$$
\begin{aligned}
&E^{x_2}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1) < H^{\mathrm{g}}(z,\omega_1)\}}\right] \\
&\quad + E^{x_2}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1) \le H^{\mathrm{g}}(y,\omega_1) < \infty\}}\right].
\end{aligned}
$$

The strong Markov property implies that the second term is equal to

$$
\begin{aligned}
&\sum_{w \in \partial_{\mathrm{ext}}\mathcal{C}_z(\omega_1)} E^{x_2}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(z,\omega_1)-1} \omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1) \le H^{\mathrm{g}}(y,\omega_1) < \infty,\, S_{H^{\mathrm{g}}(z,\omega_1)} = w\}}\right] \\
&\quad \times E^{w}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1) < \infty\}}\right]
\end{aligned}
\tag{3.7}
$$

In case 1, we have $H^{\mathrm{g}}(y, \omega_1) = H^{\mathrm{g}}(y, \omega_2)$ since $\overline{\mathcal{C}}_y(\omega_1) = \overline{\mathcal{C}}_y(\omega_2)$. Furthermore, noting that $\omega_1(z) \le \omega_2(z)$, one has $\overline{\mathcal{C}}_z(\omega_2) = \overline{\mathcal{C}}_z(\omega_1)$ if z is bad in ω_2, and $\overline{\mathcal{C}}_z(\omega_2) = \{z\} \subset \overline{\mathcal{C}}_z(\omega_1)$ otherwise. With these observations, the last expectation in (3.7) is bounded below uniformly in w by

$$
\exp\left\{-\sum_{v \in \overline{\mathcal{C}}_z(\omega_1)} (\omega_2(v) + 2\log 2)\right\} e^*(z_2, y, \omega_2) \ge \exp\{-v(z, \omega_1, \omega_2)\}\, e^{\mathrm{g}}(z, y, \omega_2).
$$

This together with (3.7) shows that

$$
\begin{aligned}
\widetilde{e}^*(x_2, y, \omega_1, \omega_2) \ge{}& E^{x_2}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_1(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1) < H^{\mathrm{g}}(z,\omega_1)\}}\right] \\
&+ E^{x_2}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(z,\omega_1)-1} \omega_1(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1) \le H^{\mathrm{g}}(y,\omega_1) < \infty\}}\right] \\
&\times \exp\{-v(z, \omega_1, \omega_2)\}\, e^{\mathrm{g}}(z, y, \omega_2),
\end{aligned}
$$

where we used that ω_1 and ω_2 coincide outside z. On the other hand, by using the strong Markov property again,

$$\begin{aligned}
&E^{x_2}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1}\omega_1(S_k)\Bigg\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1)\le H^{\mathrm{g}}(y,\omega_1)<\infty\}}\Bigg]\\
&\le E^{x_2}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H^{\mathrm{g}}(z,\omega_1)-1}\omega_1(S_k)\Bigg\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1)\le H^{\mathrm{g}}(y,\omega_1)<\infty\}}\Bigg]e^{\mathrm{g}}(z,y,\omega_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\widetilde{e}^*(x_2,y,\omega_1,\omega_2)}{e^*(x_1,y,\omega_1)}\\
&\ge e^*(x_1,y,\omega_1)^{-1}E^{x_2}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1}\omega_1(S_k)\Bigg\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1)<H^{\mathrm{g}}(z,\omega_1)\}}\Bigg]\\
&\quad+\exp\{-v(z,\omega_1,\omega_2)\}\,\frac{e^{\mathrm{g}}(z,y,\omega_2)}{e^{\mathrm{g}}(z,y,\omega_1)}\\
&\quad\times e^*(x_1,y,\omega_1)^{-1}E^{x_2}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1}\omega_1(S_k)\Bigg\}\mathbf{1}_{\{H^{\mathrm{g}}(z,\omega_1)\le H^{\mathrm{g}}(y,\omega_1)<\infty\}}\Bigg].
\end{aligned}\tag{3.8}$$

In particular, bound (3.2) immediately follows by only using the first term of the right side.

For bound (3.3), note that $e^{\mathrm{g}}(z,y,\omega_2)/e^{\mathrm{g}}(z,y,\omega_1)\le 1$. This is because $\omega_1(x)\le\omega_2(x)$ for $x\in\mathbb{Z}^2$ and $\overline{\mathcal{C}}_y(\omega_1)=\overline{\mathcal{C}}_y(\omega_2)$. Thus, by (3.8),

$$\frac{\widetilde{e}^*(x_2,y,\omega_1,\omega_2)}{e^*(x_1,y,\omega_1)}\ge\exp\{-v(z,\omega_1,\omega_2)\}\,\frac{e^{\mathrm{g}}(z,y,\omega_2)}{e^{\mathrm{g}}(z,y,\omega_1)},\tag{3.9}$$

and it remains to estimate the last fraction. For $\omega\in\Omega$, we use the strong Markov property to obtain

$$\begin{aligned}
&e^{\mathrm{g}}(z,y,\omega)\\
&\le\max_{z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)}E^{z'}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega)-1}\omega(S_k)\Bigg\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega)<H_2^{\mathrm{g}}(z,\omega)\}}\Bigg]\\
&\quad+e^{\mathrm{g}}(z,y,\omega)\max_{z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)}E^{z'}\Bigg[\exp\Bigg\{-\sum_{k=0}^{H_2^{\mathrm{g}}(z,\omega)-1}\omega(S_k)\Bigg\}\mathbf{1}_{\{H_2^{\mathrm{g}}(z,\omega)\le H^{\mathrm{g}}(y,\omega)<\infty\}}\Bigg],
\end{aligned}$$

where $H_2^{\mathrm{g}}(z,\omega)$ is the time of the second visit of y. It follows that

$$\begin{aligned}
&e^{\mathrm{g}}(z,y,\omega)\\
&\leq \frac{\max_{z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)} E^{z'}\Big[\exp\big\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega)-1}\omega(S_k)\big\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega)<H_2^{\mathrm{g}}(z,\omega)\}}\Big]}{1-\max_{z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)} E^{z'}\Big[\exp\big\{-\sum_{k=0}^{H_2^{\mathrm{g}}(z,\omega)-1}\omega(S_k)\big\}\mathbf{1}_{\{H_2^{\mathrm{g}}(z,\omega)\leq H^{\mathrm{g}}(y,\omega)<\infty\}}\Big]}\\
&\leq (1-e^{-\kappa})^{-1}\max_{z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)} E^{z'}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega)-1}\omega(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega)<H_2^{\mathrm{g}}(z,\omega)\}}\right].
\end{aligned}\tag{3.10}$$

On the other hand, the definition of $e^{\mathrm{g}}(z,y,\omega)$ implies that for $z'\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega)$,

$$e^{\mathrm{g}}(z,y,\omega)\geq E^{z'}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega)-1}\omega(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega)<H_2^{\mathrm{g}}(z,\omega)\}}\right].$$

Hence, taking $\omega=\omega_2$, one has

$$\begin{aligned}
e^{\mathrm{g}}(z,y,\omega_2)&\geq \exp\{-v(z,\omega_1,\omega_2)\}\\
&\quad\times\max_{z''\in\partial_{\mathrm{ext}}\mathcal{C}_z(\omega_1)} E^{z''}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1}\omega_1(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1)<H_2^{\mathrm{g}}(z,\omega_1)\}}\right].
\end{aligned}$$

This, combined with (3.10) for $\omega=\omega_1$, enables us to show that

$$\frac{e^{\mathrm{g}}(z,y,\omega_2)}{e^{\mathrm{g}}(z,y,\omega_1)}\geq (1-e^{-\kappa})\exp\{-v(z,\omega_1,\omega_2)\},\tag{3.11}$$

and bound (3.3) follows.

We next treat case 2. Suppose that $\mathrm{dist}(\overline{\mathcal{C}}_0(\omega_1),\overline{\mathcal{C}}_z(\omega_1))=0$. In this case, we can take an $x_0\in\partial_{\mathrm{ext}}\mathcal{C}_0(\omega_1)\cap\partial_{\mathrm{ext}}\mathcal{C}_z(\omega_1)$ and $\overline{\mathcal{C}}_y(\omega_2)\subset\overline{\mathcal{C}}_y(\omega_1)$. Hence,

$$\begin{aligned}
\widetilde{e}^{\mathrm{g}}(0,y,\omega_1,\omega_2)&\geq \widetilde{e}^*(x_0,y,\omega_1,\omega_2)\\
&\geq E^{x_0}\left[\exp\left\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_2)-1}\omega_2(S_k)\right\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_2)<\infty\}}\right]\\
&\geq \exp\{-v(z,\omega_1,\omega_2)\}\,e^{\mathrm{g}}(z,y,\omega_2).
\end{aligned}$$

Furthermore, we use $x_0\in\partial_{\mathrm{ext}}\mathcal{C}_0(\omega_1)\cap\partial_{\mathrm{ext}}\mathcal{C}_z(\omega_1)$ again to obtain

$$e^{\mathrm{g}}(z,y,\omega_1)\geq e^*(x_0,y,\omega_1)\geq \exp\{-v(0,\omega_1,\omega_2)\}\,e^{\mathrm{g}}(0,y,\omega_1).$$

Therefore,

$$\frac{\widetilde{e}^{\mathrm{g}}(0,y,\omega_1,\omega_2)}{e^{\mathrm{g}}(0,y,\omega_1)}\geq \exp\{-(v(0,\omega_1,\omega_2)+v(z,\omega_1,\omega_2))\}\,\frac{e^{\mathrm{g}}(z,y,\omega_2)}{e^{\mathrm{g}}(z,y,\omega_1)}.\tag{3.12}$$

If $\mathrm{dist}(\overline{\mathcal{C}}_y(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) = 0$, then there exists $z_0 \in \partial_{\mathrm{ext}}\mathcal{C}_y \cap \partial_{\mathrm{ext}}\mathcal{C}_z$. This means that

$$e^{\mathrm{g}}(z, y, \omega_1) \geq e^*(z_0, y, \omega_1) = 1,$$

which proves $e^{\mathrm{g}}(z, y, \omega_1) = 1$. Thus, the last fraction of (3.12) is equal to

$$\begin{aligned} &e^{\mathrm{g}}(z, y, \omega_2) \\ &\geq \begin{cases} E^{z_0}\Big[\exp\big\{-\sum_{k=0}^{H^{\mathrm{g}}(y,\omega_1)-1} \omega_2(S_k)\big\}\mathbf{1}_{\{H^{\mathrm{g}}(y,\omega_1)<\infty\}}\Big] = 1, & \text{if } z \text{ is bad in } \omega_2, \\ \exp\{-v(z, \omega_1, \omega_2)\}, & \text{otherwise.} \end{cases} \end{aligned}$$

This together with (3.12) derives bound (3.4).

On the other hand, if $\mathrm{dist}(\overline{\mathcal{C}}_y(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) \geq 1$, then bound (3.4) immediately follows from (3.11).

Finally, suppose that

$$\mathrm{dist}(\overline{\mathcal{C}}_0(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) \geq 1 \text{ and } \mathrm{dist}(\overline{\mathcal{C}}_y(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) = 0.$$

By (3.9), we have

$$\frac{\widetilde{e}^{\mathrm{g}}(0, y, \omega_1, \omega_2)}{e^{\mathrm{g}}(0, y, \omega_1)} \geq \exp\{-v(z, \omega_1, \omega_2)\} \frac{e^{\mathrm{g}}(z, y, \omega_2)}{e^{\mathrm{g}}(z, y, \omega_1)}.$$

We now assume $\mathrm{dist}(\overline{\mathcal{C}}_y(\omega_1), \overline{\mathcal{C}}_z(\omega_1)) = 0$, so that the same argument below (3.12) shows that the last fraction is bigger than or equal to $\exp\{-v(z, \omega_1, \omega_2)\}$. Therefore, (3.4) follows. □

Proof of Theorem 3.3. If $\mathrm{dist}(z, \mathcal{C}_0(\omega_1)) \geq 1$ and $\mathrm{dist}(z, \mathcal{C}_y(\omega_1)) \geq 1$, then we have $\partial_{\mathrm{ext}}\mathcal{C}_0(\omega_1) = \partial_{\mathrm{ext}}\mathcal{C}_0(\omega_2)$ and $\partial_{\mathrm{ext}}\mathcal{C}_y(\omega_1) = \partial_{\mathrm{ext}}\mathcal{C}_y(\omega_2)$. Hence,

$$\widetilde{a}^{\mathrm{g}}(0, y, \omega_2) - \widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) = 0.$$

Suppose that $\mathrm{dist}(z, \mathcal{C}_0(\omega_1)) = 0$ or $\mathrm{dist}(z, \mathcal{C}_y(\omega_1)) = 0$. Then,

$$e^{\mathrm{g}}(0, y, \omega_2) \geq e^{-(v(0,\omega_1,\omega_2)+v(y,\omega_1,\omega_2))} e^{\mathrm{g}}(0, y, \omega_1, \omega_2),$$

which proves

$$\widetilde{a}^{\mathrm{g}}(0, y, \omega_2) - \widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) \leq v(0, \omega_1, \omega_2) + v(y, \omega_1, \omega_2).$$

Furthermore, it is clear that $\widetilde{a}^{\mathrm{g}}(0, y, \omega_2) - \widetilde{a}^{\mathrm{g}}(0, y, \omega_1, \omega_2) \geq 0$ holds because $\overline{C}_0(\omega_2) \subset \overline{C}_0(\omega_1)$ or $\overline{C}_y(\omega_2) \subset \overline{C}_y(\omega_1)$. □

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